数理解析研究所講究録
534巻 1984年 151-160



# On a decomposition of a connected graph

KIYOSHI AOKI (青木 清)
*Niigata College of Pharmacy*
SEIICHI WATABE (渡部清一)
*Niigata Academy of Medical Technology*

## 1. INTRODUCTION

The total number of nonidentical graphs of order p with the same vertex set V is $2^{p(p-1)/2}$ since there are p(p-1)/2 distinct pairs of vertices. It is not difficult to see that the number of non-identical (p,q)-graphs is $\binom{p(p-1)/2}{q}$ for fixed integers p and q with $p \geqq 1$ and $0 \leqq q \leqq \binom{p}{2}$. But the corresponding problem of determining the number of nonisomorphic (p,q)-graph, for fixed integers p and q, is considerably more difficult. Also, many papers give characterizations of the degree sequence. Some characterizations of the degree sequence with the unique realization are given by S.L.Hakimi [5] and P.Erdös [6]. It is often the case that two graphs have the same structure, differing only in the way their vertices and edges are labeled or only in the way they are represented geometrically. One of the most common problems in graph theory deals with the equivalence relation on graphs. It still remains an unsolved problem to discover an useful characterization of isomorphic graphs, although the relation "isomorphic to" devides the collection of all graphs into equivalence classes.

In our paper [12], we have been investigated the characterization of nonisomorphic graphs using the concept of the incident degree sequence. In this paper, we shall introduce the new concepts which are the Hamilton walk [3] and the decomposition by the Hamilton walks. And we shall characterize the nonisomorphic graphs by using the degree sequence and these concepts. We can determine whether two graphs with the same degree sequence is isomorphic or not. Some aspects of the decomposition by the Hamilton walks are investigated.

## 2. DEFINITIONS AND PRELIMINARIES

A graph [8] G is an ordered pair of disjoint set (V,E) such that E is a subset of the set of unordered pairs of V. Unless it is explecitly stated otherwise, we consider only finite graphs, that is V and E are always finite. The set V is the set of vertices and E is the set of edges. If G is a graph then V=V(G) is the vertex set of G and E=E(G) is the edge set of G. The order of G is the number of vertices and it is denoted by $|V(G)|$. The size of G is the number of edges and it is denoted by $|E(G)|$. For a graph G, if $|V(G)| = p$ and $|E(G)|=q$, then G is called a (p,q)-graph. A graph of order n and size $\binom{n}{2}$ is called a complete n-graph and is denoted by $K_n$. A graph G is isomorphic [2] to a graph H if there exists a one-to-one mapping f, called an isomorphism, from V(G) onto V(H) such that f preserves adjacency, that is, $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. If G is isomorphic to H, then we say that G and H are isomorphic and write $G \cong H$. The degree [8] of a vertex $v_i$ of a graph G is the cardinality of the set of vertices adjacent to $v_i$; it is denoted by $d(v_i)$ or $d_i$ simply. The minimum degree of the vertices of a graph G is denoted by $\delta(G)$ and maximum degree by $\Delta(G)$. If $V(G) = \{v_1,...,v_n\}$ then a sequence $d(v_1),...,d(v_n)$ is called a degree sequence of G. Usually we order the vertices in such a way that the degree sequence obtained in this way is nondecreasing. A walk [1] is an alternating sequence of vertices and edges, say $x_0,e_1,x_1,...,e_k,x_k$ where $e_i=x_{i-1}x_i$ and $0<i<k$. This walk is usually denoted by $x_0x_1...x_k$. Note that a path is a walk with distinct vertices. A graph is connected if for every pair $\{u,v\}$ of distinct vertices there is a path from u to v. A cycle containing all the vertices of a graph G is said to be a Hamilton cycle [11] of the graph. No efficient algorithm is known for constructing a Hamilton cycle. A Hamilton path of a graph is a path containing all the vertices of the graph. And we shall define especially that a walk with all the vertices of a graph G is called a Hamilton walk in this paper. A compact orientable 2-manifold [4] is a surface that may be thought of as a sphere on which a number of handles has been placed. The number of handles is reffered as the genus of the surface. By the genus [7][9] $\gamma(G)$ of a graph G is meant the smallest genus of all surface (compact orientable 2-manifolds) on which G can be embedded.

## 3. RESULTS

As above mentioned, we introduce the concept which is the Hamilton walk. Let G be a connected (p,q)-graph and let $g_0$ be the degree sequence of nondecreasing order. Next, we shall construct the

Hamilton walk as follows and let us consider the another sequence $(a_1,\dots,a_m)$.

$P_1$: a vertex of the minimum degree $\delta(G)=a_1$.

$P_2$: a vertex whose degree is the minimum degree $a_2$ among the adjacent vertices with $P_1$.

$P_3$: a vertex whose degree is the minimum degree $a_3$ among the adjacent vertices with $P_2$ except for $P_1$.

Let continue this labeling until all the vertices of G are labeled, where edges and vertices must avoid overlapping as possible and we give priority to the vertex of the low degree. According to the above mentioned method, we can get the degree sequence $g_1=(a_1,\dots,a_s)$ and the corresponding to the vertex sequence $G_1=(P_1,\dots,P_s)$ which is called the first Hamilton walk. If there are several sequences $g_1^{(1)},\dots,g_1^{(j)}$, then we define $g_1$ again as the minimal sequence from among these degree sequences by the lexicographical order. The vertex sequence corresponding to $g_1$ is denoted by $G_1=(P_1,\dots,P_s)$ again. And if there are several sequences as same as $g_1$, then we construct the second sequence $g_2$ based on $g_1$. And we choose the base of the minimal second sequence $g_2$ as $g_1$ again. Now, we construct the second Hamilton walk $G_2$ as follows; let $Q_1$ be the initial vertex $P_1$ of $G_1$. And the edge which is used in $G_1$ should be used in $G_2$ as possible. If there is no edge which is not used in $G_1$ then we construct the sequence that we give priority to the vertex of the low index in $G_1$. Then we can construct the second Hamilton walk $G_2=(Q_1,\dots,Q_t)$ and $g_2=(b_1,\dots,b_t)$ which is the corresponding sequence to $G_2$ where $b_k$ is equal to the index number m if $Q_k=P_m$. And if there are several sequences as $g_2$, then we may define $g_2$ as the minimal sequence from among these sequences by lexicographical order. Similarly, we can get the another Hamilton walks $G_3,\ldots,G_\alpha$ and the corresponding sequences $g_3,\ldots,g_\alpha$ respectively. This is denoted by $g=(g_0,\ldots,g_\alpha)$ is the decomposition by the Hamilton walks of G in this paper. And a vector space whose generators are $G_1,\ldots,G_\alpha$, denoted by $\mathcal{P}(G)$ [10], is called a path space of G in this paper.

Now, we can get the following theorem.

THEOREM 1. *The correspondence $\varphi$ from G to the decomposition $g=(g_0,\ldots,g_\alpha)$ of G is a one-to-one mapping if and only if there exists no graph except for G which has the same degree condition with G for the extension of adding edges suitably to $\dot{\bigcup}_{i=1}^{\alpha} G_i$.*

*Proof.* Let $\{G_i\}$ and $\{H_i\}$, where $i=1,\ldots,\alpha$, be the set of the Hamilton walks in G and H respectively. Assume that there exists a mapping $\varphi$ from G and H to $g=(g_0,\ldots,g_\alpha)$ and $h=(h_0,\ldots,h_\alpha)$ respectively. At first, we shall show that if g=h, then $\dot{\bigcup}_{i=1}^{\alpha} G_i \cong \dot{\bigcup}_{i=1}^{\alpha} H_i$. We can get $G_1 \cong H_1$ by mapping the k-th vertex of $G_1$ to the k-th vertex of $H_1$ and the k-th edge of $G_1$ to the k-th edge of $H_1$. Similarly, we can get $G_i \cong H_i$ successively. Thus, it is clear that g=h implies $g_i=h_i$ for all i. By composing their isomorphisms we can get $\dot{\bigcup}_{i=i}^{\alpha} G_i$ and $\dot{\bigcup}_{i=1}^{\alpha} H_i$ are isomorphic. Since we can identify $\dot{\bigcup}_{i=1}^{\alpha} G_i \cong \dot{\bigcup}_{i=1}^{\alpha} H_i$, we get $G \cong H$ by the assumption that for the extension of adding edges suitably to $\dot{\bigcup}_{i=1}^{\alpha} G_i$ there exists no graph except for G which has the same degree condition with G. Therefore, g=h implies $G \cong H$. Conversely, assume that g=h implies $G \cong H$. As above mentioned, we can get $\dot{\bigcup}_{i=1}^{\alpha} G_i \cong \dot{\bigcup}_{i=1}^{\alpha} H_i$ and $G \cong H$. Thus, we can get the above condition. Therefore, the statement is correct.

Next, as a consequence of theorem 1, we can get the classification of (p,q)-graphs with respect to p and q where $p=1,\dots,6$ [1]. Now we consider the classification for the connected graphs of order $p\leqq 6$. It is obvious that for the disconnected graphs we may consider the classification of the connected components.

THEOREM 2. *Let G be a connected graph of order $p(\leqq 6)$ and let $g_0$ be the degree sequence of G. Then a correspondence $\varphi_0$ from G to $g_0$ is not a one-to-one mapping at the following cases; if $p=5$, then $g_0=(2^3 3^2)$ and if $p=6$, then $g_0=(2^4 3^2)$, $(2^2 3^4)$, $(2^3 3^2 4)$, $(2^4 4^2)$ $(23^4 4)$, $(2^2 3^2 4^2)$, $(23^3 45)$, $(3^4 4^2)$, $(23^2 4^3)$, $(3^3 4^2 5)$, $(3^2 4^4)$, $(3^6)$ and $(4^6)$.*

*Proof.* According to above mentioned classifications [1], it is easy to obtain the statement. Clearly, there exist the graphs that they have the same degree sequence but they are not isomorphic mutually.

Next, we get the following theorem.

THEOREM 3. *Let G be a connected graph of order 5. A correspondence $\varphi_1$ from G to $(g_0, g_1)$ is a one-to-one mapping.*

*Proof.* By the deffinition of $g_1$ it is obvious. It is enough to consider the case of $g_0=(2^3 3^2)$ by theorem 2. For the nonisomorphic graphs with the same $g_0=(2^3 3^2)$, their first Hamilton walks are all distinct. Therefore, $\varphi_1$ is a one-to-one mapping.

At the case of order 6 we can get the following theorem.

THEOREM 4. *Let G be a connected graph of order 6. Then a correspondence* $\mathcal{G}_1$ *from G to* $(g_0, g_1)$ *is not a one-to-one mapping at the only following cases;* $g_0=(2^2 3^4)$, $(23^4 4)$, $(3^6)$ *and* $(4^6)$.

*Proof.* In fact, at the case of $(2^2 3^4)$ there are four distinct kinds of graphs and their first Hamilton walks are equal to (233233), (233233), (232333) and (223333) respectively. At the case of $(23^4 4)$ there are three distinct kinds of graphs and $g_1$ is equal to $g_0$ respectively. At the case of $(3^6)$ and $(4^6)$ there are two distinct kinds of graphs respectively and $g_1$ is equal to $g_0$ respectively. For the other cases of theorem 2 it is similar to theorem 3.

According to above mentioned facts, we can get the following theorem.

THEOREM 5. *Let G be a connected graph of order 6. Then a correspondence* $\mathcal{G}_2$ *from G to* $(g_0, g_1, g_2)$ *is a one-to-one mapping.*

*Proof.* It is enough to show that the graphs which have the same first sequence $g_1$ have the distinct second sequence $g_2$ mutually. Namely, we must consider the following cases; $g_0=(2^2 3^4)$, $(23^4 4)$, $(3^6)$ and $(4^6)$. These graphs have the same $g_1$. Now, let us consider the second sequences for their graphs. By the definition of $g_2$ we can get the following; for $g_0=(2^2 3^4)$, we get (162354) and (163254) respectively. For $g_0=(23^4 4)$, we get (154623), (162354) and (163254) respectively. For $g_0=(3^6)$, we get (132546) and (1436125) respectively. For $g_0=(4^6)$, we get (1351624) and (135264) respectively. Obviously, in each case they are distinct mutually. Therefore, a correspondence $\varphi_2$ from G to $(g_0, g_1, g_2)$ is a one-to-one mapping.

Next, when there exists a one-to-one mapping from G to $(g_0,\ldots,g_\alpha)$ for some $\alpha$, $\alpha$ is called the dimension of the path space $\mathcal{P}(G)$, denoted by $\dim(\mathcal{P}(G))$ in this paper. Then we get the following theorems.

THEOREM 6. *Let G be a cycle or the complete graph of order p. Then $\dim(\mathcal{P}(G))$ is equal to 1.*

*Proof.* If G is a cycle or the cmplete graph of order p, then we get $g_0=g_1=(2^p)$ or $g_0=g_1=((p-1)^p)$ respectively. By the definition of the dimension of $\mathcal{P}(G)$ and theorem 1 we can get the statement.

THEOREM 7. *Let G be a connected 3-regular graph of order p. Then $\dim(\mathcal{P}(G))$ is equal to 2.*

*Proof.* As a consequence of theorem 1, we can get the statement easily.

Furthermore, we can get the following statements with respect to the genus.

LEMMA 8.1 *Let G be a Hamiltonian graph of an even order p and let C be a Hamilton cycle of G. Then G is embeddable on the surface with the genus $\{(p-4)/4\}$ (the smallest integer not less than $(p-4)/4$ [1]) if and only if the number of edges which are crossing mutually in the interior region of C on the sphere is equal to $p/2$.*

*Proof.* Assume that p/2 edges of G in the interior region of C are crossing mutually. Then we can draw on the sphere two edges of them which are not cross to each other. And also, it is obvious that by attaching a handle to the sphere we can put off the crossing number at least one. Thus, it is enough to attach the half of $p/2-2=(p-4)/2$ handles to the sphere. That is, we can get the genus of G is $\{(p-4)/4\}$. According to the definition of the genus, the inverse is obvious.

LEMMA 8.2 *Let G be a Hamiltonian graph of an even order p and let C be a Hamilton cycle of G. If the chords of C on the sphere are crossing mutually, then there exists the sequence* $g_k=(b_1,b_2,b_3,b_4,b_5,b_6,b_7,b_8,\ldots)$ *where* $b_1<b_5<b_8<b_4$, $b_2<b_6<b_7<b_3$, $b_1<b_2<b_3$, $b_3>b_4>b_5$, $b_5<b_6<b_7$ *and* $b_7>b_8>b_9$ *for some k.*

*Proof.* By the definition of $b_k$ and the index we can get the statement.

THEOREM 8. *Let G be a Hamiltonian graph of an even order p. Then the genus of G is equal to* $\{(p-4)/4\}$ *if and only if there exists the sequence* $g_k=(b_{i_1},b_{i_2},b_{i_3},\ldots)$ *for some k where* $b_{i_1}<b_{i_5}<b_{i_8}<b_{i_4}$, $b_{i_2}<b_{i_6}<b_{i_7}<b_{i_3}$, $b_{i_1}<b_{i_2}<b_{i_3}$, $b_{i_3}>b_{i_4}>b_{i_5}$, $b_{i_5}<b_{i_6}<b_{i_7}$, $b_{i_7}>b_{i_8}>b_{i_9}$ *and* $Q_{i_1}Q_{i_2}$, $Q_{i_3}Q_{i_4}$, $Q_{i_5}Q_{i_6}$ *and* $Q_{i_7}Q_{i_8}$ *are edges of G.*

*Proof.* Let us consider all the Hamilton walks of G. For some k we can get the vertex sequence $Q_{i_1},\ldots,Q_{i_p}$ which satisfies the same condition of lemma 8.1, the existence of such the walk is the necessary and sufficient condition for the genus of G is $\{(p-4)/4\}$. Therefore, as a consequence of lemma 8.1 and lemma 8.2, we can get the statement.

# REFERENCES

[1] F.HARARY, Graph Theory, Addison-Wesley, 1969.

[2] M.BEHZAD, G.CHARTRAND AND L.L.FOSTER, Graphs & Digraphs, Prindle Weber & Schmidt, 1979.

[3] R.J.WILSON, Introduction to Graph Theory, Academic Press, 1972.

[4] A.T.WHITE, Graphs,Groups,and Surfaces, North-Holland, 1973.

[5] S.L.HAKIMI, On the realizability of a set of integers as degrees of the vertices of a graph, J.SIAM Appl.Math.10 (1962), 496-506.

[6] P.ERDÖS, Graphs with prescribed degrees of vertices, Mat.Lapok 11 (1960), 264-274.

[7] P.C.KAINEN, Some recent results in topological graph theory, Springer-Verlag, Lecture note in Math. 406, 1974.

[8] B.BOLLOBAS, Graph Theory, Springer-Verlag, 1979.

[9] J.W.T.YOUNGS, Minimum imbeddings and the genus of a graph, J.Math Mech.12 (1963), 303-315.

[10] J.E.GRAVER, The imbedding index of a graph, J.Comb.Theory B27 ( 1979), 151-159.

[11] L.W.BEINEKE AND E.F.SCHMEICHEL, Degrees and cycles in graphs, Ann N.Y. Acad. Sciences (1979), 64-70.

[12] S.WATABE AND K.AOKI, On the degree sequence of a connected graph, Bull.Niigata College of Pharmacy 2 (1982), 27-32.